

HARMONIC MEASURES ON COVERS OF COMPACT SURFACES OF NONPOSITIVE CURVATURE

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ABSTRACT. Let M be the universal cover of a compact nonflat surface N of nonpositive curvature. We show that on the average the Brownian motion on M behaves similarly to the Brownian motion on negatively curved manifolds. We use this to prove that harmonic measures on the sphere at infinity have positive Hausdorff dimension and if the geodesic flow on N is ergodic then the harmonic and geodesic measure classes at infinity are singular unless the curvature is constant.

1. INTRODUCTION

Let M be a smooth complete, simply connected d -dimensional Riemannian manifold with all sectional curvatures nonpositive. There is a natural geometric compactification of M by the sphere at infinity $S(\infty)$ which is the set of classes of asymptotic geodesics. Let Δ be the Laplace-Beltrami operator on M ; then C^2 -functions on M satisfying $\Delta h = 0$ are called harmonic. It is known that if M has bounded geometry, geodesics diverge exponentially fast, and $\Delta \rho \geq \text{const} > 0$ with $\rho(x) = \text{dist}(x_0, x)$, then there is a one-to-one correspondence between continuous functions f on $S(\infty)$ and harmonic functions h_f having a continuous extension to $S(\infty)$ given by the formula (see, for instance, [Ki]),

$$(1.1) \quad h_f(x) = \int_{S(\infty)} f(\zeta) P(x, d\zeta)$$

where $P(x, \cdot)$, $x \in M$, $P(x, S(\infty)) = 1$, are the so-called harmonic measures. Let $S_x M$ be the sphere of unit vectors from the tangent space $T_x M$. Define $\Phi_x: S_x M \rightarrow S(\infty)$ which maps a vector $\zeta \in S_x M$ to the end at ∞ of the geodesic with initial velocity ζ . It turns out (see [Ka2, L]) that, in general, the measure $\nu_x = \Phi_x^{-1} P(x, \cdot)$ on $S_x M$ is singular with respect to the Lebesgue measure on $S_x M$. But still under the conditions of [Ki], which hold true if all sectional curvatures of M are bounded from above and below by two negative constants, the Hausdorff dimension of ν_x is positive (see [KL]).

Of course, these conditions fail to be true without some kind of negative curvature assumption. Nevertheless, if M is the universal cover of a compact

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rank-one manifold N of nonpositive curvature with the metric on M lifted from N , then by [B] relation (1.1) still gives a one-to-one correspondence between harmonic functions h_f on M and their continuous boundary values f on $S(\infty)$, and so in this case as well one can speak of harmonic measures. The natural question whether these harmonic measures (i.e., the corresponding measures $\Phi_x^{-1}P(x, \cdot)$) have positive Hausdorff dimensions will be answered positively in this paper only in the two-dimensional case. The reason is that our argument uses global exponential divergence of geodesics which in dimension two is the same as the exponential growth of volume on geodesic spheres. The other condition $\Delta\rho > \text{const} > 0$ which was used in [KL] to derive the positivity of the Hausdorff dimensions of harmonic measures does not hold true everywhere in our circumstances. However, the argument actually uses only the positivity of the radial drift of the Brownian motion $X(t)$ on M and this holds true in a certain average sense on covers of compact manifolds of nonpositive curvature with all sectional curvatures negative at a point. These ideas enable us to employ the machinery from [KL] and prove the positivity of the Hausdorff dimensions, as well as to modify the method of [L] in order to prove the following extension of [Ka2 and L].

If M is the universal cover of a compact nonflat surface of nonpositive curvature N and the geodesic flow on N is ergodic, then the measure $\nu_x = \Phi_x^{-1}P(x, \cdot)$ is equivalent to the Lebesgue measure on S_xM if and only if N has constant curvature. Concerning the ergodicity assumption note that though the proof in [BB] contains a gap where they use an unproven assertion from [P], it is believed that the geodesic flow on a nonflat surface of nonpositive curvature is ergodic.

Our results admit the following purely geometric interpretation. Let M be the universal cover of a nonflat surface N of nonpositive curvature. Then M has a standard representation as the Poincaré open unit disk D with boundary $S(\infty)$ and center \mathcal{O} . By the uniformization theorem, M is conformally equivalent to D with the Euclidean metric; the Brownian motion on M differs from the usual (Euclidean) Brownian motion in D by a time change, and hence harmonic measures for these Brownian motions are the same. In particular, $P(\mathcal{O}, \cdot)$ is the normalized Lebesgue measure on $S(\infty)$. For each $\gamma \in S(\infty)$, let $\xi = \xi_\gamma \in S_{\mathcal{O}}D$ be the direction of the geodesic in the metric inherited from M connecting γ and \mathcal{O} and let $\eta(\xi) \in S_{\mathcal{O}}D$ be the direction of the Euclidean radius connecting γ and \mathcal{O} . Then $\eta: S_{\mathcal{O}}D \rightarrow S_{\mathcal{O}}D$ is a homeomorphism of the unit circle. The singularity of harmonic measures means that η maps the Lebesgue measure to a singular measure (assuming that the geodesic flow on N is ergodic) unless the curvature of M is constant; however, the Hausdorff dimension of this measure is positive.

By the Harnack inequality it is easy to see (see, e.g., [Ki]) that all harmonic measures are equivalent, and so one has a harmonic measure class. The map Φ_x projects the normalized Lebesgue measure m_x on S_xM to a probability measure μ_x on $S(\infty)$. If M is the universal cover of a compact manifold N of negative curvature, then it is well known that measures μ_x for all $x \in M$ are equivalent, and so one has a geodesic measure class. In the nonpositive curvature case whether or not the measures μ_x are equivalent remains an open question since [BB] contains a gap in the proof though it is believed to be true,

at least in the two-dimensional case. This is equivalent to the ergodicity of the geodesic flow on N . We will prove here under this condition that all harmonic measures $P(x, \cdot)$, $x \in M$, are singular with respect to all μ_y , $y \in M$, unless the curvature is constant.

2. GEOMETRIC PRELIMINARIES

We assume that M is the universal cover of a compact manifold N of non-positive sectional curvature. By the Cartan-Hadamard theorem the exponential map $\text{Exp}_x: T_x M \rightarrow M$ based at any point $x \in M$ is a diffeomorphism of $T_x M$ onto M . Hence there is a global system of geodesic polar coordinates which assigns to each $z \in M$ the pair (r, ξ) where $r = \text{dist}(x, z)$, $\xi \in S_x M$, $z = \text{Exp}_x(r\xi)$, and dist denotes the Riemannian distance. In these coordinates the Laplace-Beltrami operator can be written in the form

$$(2.1) \quad \Delta = \frac{\partial^2}{\partial r^2} + Q_x(r, \xi) \frac{\partial}{\partial r} + \Delta_\xi$$

where Δ_ξ does not contain derivatives in r , and

$$(2.2) \quad Q_x(r, \xi) = \frac{\partial}{\partial r}(\log A_x(r, \xi)) = \Delta \rho_x(r) > 0.$$

Here ρ_x is the distance function from x and $A_x(r, \xi)$ is the $(d-1)$ -dimensional volume element on the geodesic sphere $S_x(r)$ of radius r centered at x (see [IW, §VI.5] and [Ch, p. 67]). Note that $Q(r, \xi)$ is the trace of the operator of the second fundamental form of the sphere $S_x(r)$ (see [G, Chapter 3]). Since M is the universal cover of a compact manifold N , the Ricci curvature of M is bounded below. Hence, by Bishop's comparison theorem (see [Ch, pp. 71–73]), $Q_x(r, \xi)$ is uniformly bounded from above for all $r \geq \delta > 0$. Let $B_x(r)$ denote the ball of radius r centered at x in M or in N .

2.1. Lemma. *Let M be the universal cover of a compact manifold N of non-positive curvature of rank 1 (see [BB]). Then there exist $R_0, b_0, r_0 > 0$ such that for any $x, y \in M$ there is $z_y \in B_y(R_0)$ with the property $Q_x(r, \xi) \geq b_0$ for any $(r, \xi) \in B_{z_y}(r_0)$.*

Proof. Let us show first that there are $R_0, \tilde{b}_0 > 0$ such that for any $x, y \in M$ there exists $z_y \in B_y(R_0)$ for which $Q_x(r, \xi) \geq \tilde{b}_0$ with $(r, \xi) = z_y$. Suppose not. Then there are sequences $x_k, y_k \in M$ such that

$$(2.3) \quad Q_{x_k}(r, \xi) < \frac{1}{k}$$

for any $(r, \xi) \in B_{y_k}(k)$. Obviously $d_k = \text{dist}(x_k, y_k) \rightarrow \infty$ as $k \rightarrow \infty$ because $Q_x(r, \xi) \geq (d-1)/r$, where $d = \dim M$. Since M is the universal cover of a compact manifold, there is a sequence of isometries θ_k for which the points $\theta_k y_k$ lie in the same fundamental domain. Set $x'_k = \theta_k x_k$, $y'_k = \theta_k y_k$ and choose a subsequence $n_k \rightarrow \infty$ such that

$$(2.4) \quad \text{(i) } x'_{n_k} \rightarrow \bar{x} \in S(\infty), \quad \text{(ii) } y_{n_k} \rightarrow \bar{y} \in M.$$

The horosphere $H_{\bar{x}}(\bar{y})$ passing through \bar{y} and centered at \bar{x} is uniformly approximated in the C^2 -topology (on compact subsets) by the spheres $S_{x'_{n_k}}(d_{n_k})$, see [HI]. Let $V_{\bar{x}}(z)$ be the operator of the second fundamental form of $H_{\bar{x}}(\bar{y})$

at $z \in H_{\bar{x}}(\bar{y})$. Since the curvature of M is nonpositive, $V_{\bar{x}}(z)$ is a nonnegative selfadjoint operator. By (2.3) and (2.4), $\text{tr } V_{\bar{x}}(z) = 0$ for any $z \in H_{\bar{x}}(\bar{y})$. Hence,

$$(2.5) \quad V_{\bar{x}}(z) = 0 \quad \text{for any } z \in H_{\bar{x}}(\bar{y}).$$

It follows now from [CE] that M admits a Riemannian splitting $M = M' \times \mathbb{R}^1$ and has rank at least 2, which contradicts the assumption and proves the claim above.

Since the spheres approximate the horospheres in the C^2 -topology uniformly on compact subsets [HI], the functions $Q_x(r, \xi)$ are uniformly equicontinuous for $r > \delta$ for any $\delta > 0$. Therefore, we can choose $b_0 = \tilde{b}_0/2$ and the lemma follows. \square

2.1.a. *Remark.* Suppose that for some point $x_0 \in N$ all sectional curvatures are negative; this happens for example if N is a rank 1 surface. There are $\tilde{r} > 0$ and $a^2 > 0$ such that for any $x \in B_{x_0}(r)$ the curvature of any 2-plane at x is less than $-a^2$. Let $\pi: M \rightarrow N$ be the projection. Then in the previous lemma we can set R_0 to be the diameter of a fundamental domain, the points z_y to be the preimages of x_0 under π , $r_0 = \tilde{r}/2$, $b_0 = a(d-1) \tanh ar_0$, where $d = \dim N$. \square

The following lemma gives an exponential estimate on the divergence of geodesics in M when $\dim M = 2$. Denote by $\text{dist}_x^{(r)}$ distance on the geodesic circle $S_x(r)$.

2.2. **Lemma.** *Let $\dim M = 2$. Then there exists γ_0 , $0 < \gamma_0 < 1$, such that for any $r_2 \geq r_1 \geq 1$,*

$$\gamma_0 e^{\gamma_0(r_2-r_1)} \leq \frac{\text{dist}_x^{(r_2)}(\text{Exp}_x r_2 \xi, \text{Exp}_x r_2 \eta)}{\text{dist}_x^{(r_1)}(\text{Exp}_x r_1 \xi, \text{Exp}_x r_1 \eta)} \leq \gamma_0^{-1} e^{\gamma_0^{-1}(r_2-r_1)}$$

for all $x \in M$, $\xi, \eta \in S_x M$, provided

$$\text{dist}_x^{(r_1)}(\text{Exp}_x r_1 \xi, \text{Exp}_x r_1 \eta) \geq 2R_0.$$

(See Lemma 2.1 for the definition of R_0 .)

Proof. Put

$$C_x(r, \xi) = \{\eta \in S_x M: \text{dist}_x^{(r)}(\text{Exp}_x r \xi, \text{Exp}_x r \eta) < R_0\}$$

and

$$S_x(\xi, \rho) = \{\text{Exp}_x(\rho + s)\eta: \eta \in C_x(\rho, \xi), 0 \leq s \leq 2R_0\}.$$

Note that if the length of the geodesic circle $S_x(\rho)$ is less than $2R_0$, then the truncated sector $S_x(\xi, \rho)$ is an annulus. Put $y = \text{Exp}_x(\rho + R_0)\xi$. Clearly $B_y(R_0) \subset B_x(\rho + 2R_0) \setminus B_x(\rho)$. Note that if a curve lies outside the circle $S_x(\rho)$, then its radial projection onto $S_x(\rho)$ is shorter than the curve itself. It follows that

$$(2.6) \quad S_x(\xi, \rho) \supset B_y(R_0).$$

Set for brevity

$$\text{dist}_x^{(r_i)}(\text{Exp}_x r_i \xi, \text{Exp}_x r_i \eta) = \text{dist}_i, \quad i = 1, 2.$$

Let $m = [\text{dist}_1/2R_0] \geq 1$ and $N = [(r_2 - r_1)/2R_0]$, where $[\]$ denotes the integer part. Consider the arc of $S_x(r_1)$ with ends $\text{Exp}_x r_1 \xi$ and $\text{Exp}_x r_1 \eta$. Divide it into subarcs $a_i = a_i(r_1)$, $i = 1, 2, \dots, m$, of length $2R_0$ and the remainder $a' = a'(r_1)$ whose length Δ is less than $2R_0$. Let $a_i(r)$ denote the radial projection of a_i onto $S_x(r)$. Since the curvature is nonpositive, the lengths of each $a_i(r)$ and of $a'(r)$ are strictly increasing functions of r . By (2.6), each subarc a_i encounters at least n times balls $B_{z_y}(r_0)$ (see Lemma 2.1) as it moves radially from $S_x(r_1)$ to $S_x(r_2)$. For each such encounter there are $\zeta \in T_x M$ and r , $r_1 < r < r_2$, such that $\text{Exp}_x r \zeta$ lies in $a_i(r)$ and is one of the points z_y from Lemma 2.1. Therefore the logarithmic derivative $Q_x(r', \zeta')$ of the length element $A_x(r', \zeta')$ (see (2.2)) is at least b_0 whenever $\text{dist}(\text{Exp}_x r' \zeta', \text{Exp}_x r \zeta) \leq r_0$. Thus $\text{length}(a_i(r_2)) \geq \exp(\delta n) \cdot \text{length}(a_i(r_1))$, where $\delta > 0$ does not depend on x, r_1, r_2, ξ, η . For $r_2 - r_1 \geq 2R_0$ we have

$$\frac{\text{dist}_2}{\text{dist}_1} \geq \frac{2R_0 m \exp(\delta n) + \Delta}{2R_0 m + \Delta} > e^{\delta n/2} \geq \exp(\delta(r_2 - r_1)/4R_0).$$

Since $\text{dist}_2 > \text{dist}_1$, the lower bound of the lemma follows for $r_2 - r_1 < 2R_0$ as well, with a properly chosen γ_0 . The upper bound holds true since the curvature of M is bounded from below (see Lemma 4.2 in [BBE]). \square

2.3. Remark. For dimensions greater than 2, Lemma 2.1, Formula (2.2) and arguments similar to those in the proof of Lemma 2.2 imply the exponential growth of volume on spheres $S_x(r)$ instead of length. In this case, exponential divergence of geodesics, which we use later, fails in general.

Recall that the geodesic flow in the unit tangent bundle SN is the one-parameter group of transformations g^t which parallel translates every unit vector ξ distance t along the geodesic with initial velocity ξ . The flow g^t preserves the Liouville measure $dm_x(\xi) dm(x)$, where m is the Riemannian volume in M and N and m_x is the Lebesgue measure on the spheres $S_x N$ and $S_x M$.

2.4. Lemma. Let $\dim M = 2$ and assume that the geodesic flow g^t is ergodic. For $\xi \in S_x M$ let $A_\delta(t, x, \xi) = \{\eta \in S_x M : \text{dist}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) \leq \delta\}$. Then for any $\delta > 0$ and almost all (x, ξ) with respect to the Liouville measure

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log m_x(A_\delta(t, x, \xi)) = -\gamma_1,$$

where $\gamma_1 > 0$ is the positive Lyapunov exponent of the geodesic flow.

Proof. Let $\xi, \eta \in S_x M$. Set $d(t) = \text{dist}(\text{Exp}_x t\xi, \text{Exp}_x t\eta)$. By the second variation formula $d''(t) \leq -\min K \cdot d(t)$. In particular, $d(t)$ is convex, so that $d(t) \geq t \cdot d'(0)$, $t \geq 0$. We have

$$d'(t)d''(t) \leq -\min K \cdot d'(t)d(t),$$

and hence

$$(d'(t))^2 \leq (d'(0))^2 + (d(t))^2(-\min K).$$

Let γ_t be the geodesic connecting $x_1 = \text{Exp}_x t\xi$ and $x_2 = \text{Exp}_x t\eta$ and let $v_i \in S_{x_i} M$, $i = 1, 2$, be unit vectors perpendicular to γ_t and pointing away from x . Let $\alpha < \frac{\pi}{2}$ be the angle between $g^t \xi$ and v_1 and $\beta < \frac{\pi}{2}$ be the angle

between $g^t\eta$ and v_2 . Then $d'(t) = \sin \alpha + \sin \beta$ and hence, using $\sin s \geq \frac{2}{\pi}s$ for $0 \leq s \leq \frac{\pi}{2}$,

$$\alpha + \beta \leq \frac{\pi}{2}d'(t).$$

Thus, for $t \geq 1$,

$$\begin{aligned} \text{dist}_{SM}(g_\xi^t, g_\eta^t) &\leq (\alpha + \beta)^2 + (d(t))^2 \\ (2.7) \quad &\leq \frac{\pi^4}{4}[(d'(0))^2 + (d(t))^2 \cdot (-\min K)] + (d(t))^2 \\ &\leq 3(2 - \min K) \cdot (d(t))^2. \end{aligned}$$

Let $y(t, \eta)$ be the length of the perpendicular Jacobi field along the geodesic $\text{Exp}_x t\eta$ with initial conditions $y(0, \eta) = 0$, $y'(0, \eta) = 1$. Let $u(t, \eta) = y'(t, \eta)/y(t, \eta)$ be the geodesic curvature of the geodesic circle $S_x(t)$ at $\text{Exp}_x t\eta$. Since the curvature is nonpositive, $u(t, \eta) > 0$. Let $U(g^t\eta)$ denote the geodesic curvature of the unstable horocycle of $g^t\eta$. By [HI], the geodesic circles converge uniformly on compact sets in the C^2 -topology to the unstable horocycles, and $u(t, \eta) - U(g^t\eta) \searrow 0$ as $t \rightarrow \infty$. Fix $\varepsilon > 0$ and let $T_0 > 1$ be such that $|u(t, \eta) - U(g^t\eta)| < \varepsilon$ for any $t \geq T_0$, and $x \in M$ and $\eta \in S_x M$.

Since the function $U(\zeta)$, $\zeta \in SM$, is uniformly continuous, there is $\delta_1 > 0$ such that $|U(\zeta_1) - U(\zeta_2)| < \varepsilon$ provided $\text{dist}_{SM}(\zeta_1, \zeta_2) < \delta_1 \sqrt{3(2 - \min K)}$. Therefore, by (2.7)

$$|u(t, \eta) - u(t, \xi)| < 3\varepsilon$$

for $\xi, \eta \in S_x M$, $t \geq T_0$, $\text{dist}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) < \delta_1$. Hence

$$(2.8) \quad \left| \ln \left(\frac{y(t, \eta)}{y(T_0, \eta)} \middle/ \frac{y(t, \xi)}{y(T_0, \xi)} \right) \right| < 3\varepsilon(t - T_0).$$

By the multiplicative ergodic theorem a.e. $\xi \in SM$ satisfies

$$(2.9) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \log y(t, \xi) = \gamma_1.$$

Choose ξ to be such a vector and assume also that ξ is uniformly recurrent under the geodesic flow, i.e., for any set F of positive measure in SN ,

$$(2.10) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi_F(g^\tau \xi) d\tau = \mu(F),$$

where χ denotes the characteristic function and μ the Liouville measure. Since N is not flat, there is a point $y \in N$ where the curvature is negative. Let F be the set of tangent vectors whose foot points have the property that the curvature of N is less than $-\delta_2$ in the δ_2 -neighborhood of the point. Then each time the geodesic $\text{Exp}_x t\xi$ passes those points, a finite arc of the geodesic circle $S_x(t)$ (of length δ_2) is stretched at least by a constant factor greater than 1. Let $\eta \in A_\delta(t, x, \xi)$. Then, by (2.10), there is $T_1 > 0$ such that

$$(2.11) \quad \text{dist}_x^{(\tau)}(\text{Exp}_x \tau\xi, \text{Exp}_x \tau\eta) < \delta_1$$

for $\tau = t(1 - \varepsilon)$, $t \geq t_1$. By combining (2.8), (2.9) and (2.11) we get

$$(2.12) \quad |\log y(t, \eta) - \gamma_1 t| \leq C_1 + C_2 \varepsilon t$$

for some constants C_1, C_2 and any $\eta \in A_\delta(t, \xi)$ provided t is large enough. Note that

$$(2.13) \quad \text{dist}_x^{(t)}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) = \int_\xi^\eta y(t, \zeta) dm_x(\zeta).$$

Since N is compact, the absolute value of the curvature is bounded, and hence the geodesic curvature of all geodesic circles with radius at least 1 is bounded from above. Therefore there is $C_3 > 0$ such that

$$(2.14) \quad \text{dist}_x^{(t)}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) \leq C_3 \text{dist}(\text{Exp}_x t\xi, \text{Exp}_x t\eta)$$

if the distance in the right-hand side is at most δ_1 and $t \geq 1$. By (2.13) and (2.14), for any $\eta \in A_\delta(t, x, \xi)$ we have

$$\begin{aligned} C_3^{-1} e^{\gamma_1 t - C_1 - C_2 \varepsilon t} \int_\xi^\eta dm_x(\xi) &\leq C_3^{-1} \text{dist}_x^{(t)}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) \\ &\leq \text{dist}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) \\ &\leq \text{dist}^{(t)}(\text{Exp}_x t\xi, \text{Exp}_x t\eta) \\ &\leq e^{\gamma_1 t + C_1 + C_2 \varepsilon t} \int_\xi^\eta dm_x(\xi), \end{aligned}$$

which proves the lemma. \square

2.5. Remark. The geodesic flow on a compact surface of nonpositive curvature is ergodic if the set of those geodesics along which the curvature is identically zero has Liouville measure zero. Whether this holds true for any nonflat compact surface is an open question. We describe now an example of a surface of nonpositive but not negative curvature for which the geodesic flow is ergodic. Consider two flat tori T_1, T_2 which we view as unit squares with opposite sides identified. Remove two discs and connect their boundary circles by a “neck” of negative curvature (see Figure 1). The curvature of the resulting surface (of genus 2) is 0 on the tori and negative on the connecting neck. Obviously every geodesic that starts on one of the tori with an irrational slope will eventually enter the neck. Also every geodesic that starts in the neck will eventually leave the neck. Therefore along a.e. geodesic the curvature is not identically 0. \square

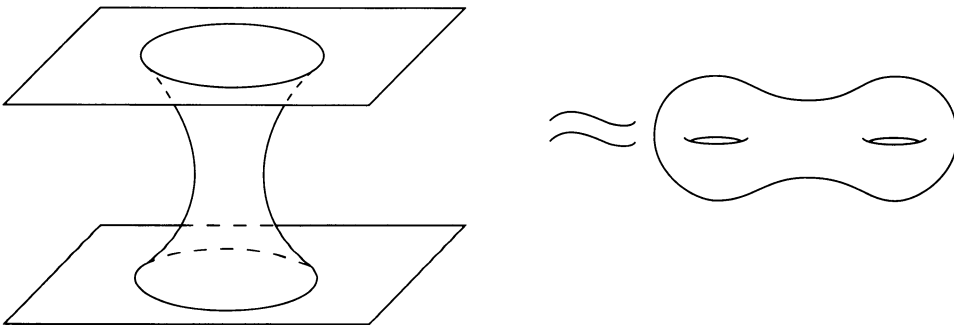


FIGURE 1

2.6. *Remark.* The absolute continuity of the weak stable horospheric foliation for M , by definition, implies that all the measures μ_x from §1 are equivalent. The ergodicity of the geodesic flow g on a rank 1 compact manifold N is equivalent to the absolute continuity of the weak stable horospheric foliation on M . Namely, if g is ergodic then almost every orbit of g visits the open subset of SN corresponding to geodesics with no parallel Jacobi fields. Hence the Lyapunov exponents of g are nonzero almost everywhere and the weak stable horospheric foliation is absolutely continuous [P]. Conversely, if the horospheric foliation is absolutely continuous, then the geodesic flow is ergodic by [P].

2.7. *Remark.* P. Eberlein informed us that the estimate of Lemma 2.2 holds true for all simply connected uniform visibility manifolds with a compact quotient in any dimension. This exponential divergence of geodesics is essentially equivalent to the manifold being a negatively curved metric space in the sense of M. Gromov. Together with Lemma 2.1 it implies that the method of the present paper works for all such manifolds. Hence the Martin boundary for such manifolds coincides with the sphere at infinity and the Hausdorff dimension of harmonic measures is positive. Note that all nonflat simply connected nonpositively curved surfaces with a compact quotient are always uniform visibility manifolds and negatively curved metric spaces in the sense of Gromov. \square

3. PROBABILISTIC LEMMAS

Recall that the Brownian motion $X(t)$ on M is a diffusion process generated by the Laplace-Beltrami operator Δ (see [IW, Chapters V and VI]). In the geodesic polar coordinates with a pole at x one can write $X(t) = (R_x(t), \Xi_x(t))$, where $\Xi_x(t) \in S_x M$ and $X(t) = \text{Exp}_x(R_x(t)\Xi_x(t))$. Let $\tau_x(r) = \inf\{t \geq 0: X(t) \in S_x(r)\}$, i.e., $\tau_x(r)$ is the first hitting time by $X(t)$ of the geodesic sphere $S_x(r)$. Let $a \wedge b = \min(a, b)$ and E_y, P_y denote the expectation and probability for the process $X(t)$ starting at y .

3.1. **Lemma.** *For all $y \in M$, $t \geq 0$ and $\delta > 0$,*

$$(3.1) \quad E_y R_x(t) \geq \text{dist}(x, y) + b_0 \delta P_y\{\tau_2 \wedge t \geq \tau_1 + \delta\},$$

where $\tau_1 = \tau_{z_y}(r_0/2)$, $\tau_2 = \inf\{s > \tau_1: X(s) \in S_{z_y}(r_0)\}$, z_y, r_0, b_0 are the same as in Lemma 2.1.

Proof. Since the Laplace-Beltrami operator Δ has the form (2.1) in geodesic polar coordinates, the radial part $R_x(t)$ of $X(t)$ satisfies the following stochastic equation

$$(3.2) \quad R_x(t) = R_x(0) + \sqrt{2}w(t) + \int_0^t Q_x(X(s)) ds,$$

where $w(t)$ is the one-dimensional Wiener process starting at zero. Furthermore,

$$(3.3) \quad \begin{aligned} R_x(t) - R_x(0) &= \sqrt{2}w(t) + \int_0^{\tau_1 \wedge t} Q_x(X(s)) ds \\ &\quad + \chi_{\tau_1 < t} \int_{\tau_1}^{\tau_2 \wedge t} Q_x(X(s)) ds + \int_{\tau_2 \wedge t}^t Q_x(X(s)) ds \\ &\geq \sqrt{2}w(t) + b_0(\tau_2 \wedge t - \tau_1)\chi_{\tau_1 < t}, \end{aligned}$$

where χ_A denotes the indicator of event A . Taking the expectation of both parts in (3.3), one gets (3.1). \square

3.2. Lemma. *There exists $\alpha > 0$ such that for any $x \in M$ with probability one*

$$(3.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \text{dist}(x, X(t)) = \lim_{t \rightarrow \infty} \frac{1}{t} R_x(t) = \alpha.$$

Proof. The existence of the limit in (3.4) is known (see [Va, §II.7]) whenever M is the universal cover of a compact manifold N . The argument goes as follows. Let $\pi: M \rightarrow N$ be the natural projection. Then πX_t is the Brownian motion on N . It preserves the normalized volume on N which is the unique probability invariant measure of πX_t , and so it can be considered as an ergodic stationary process. Let $\theta_s: \Omega \rightarrow \Omega$ be the corresponding time shift so that $\pi X(t, \theta_s \omega) = \pi X(t+s, \omega)$. Any continuous path φ_s , $0 \leq s \leq t$, on N can be uniquely lifted to M provided we fix a fundamental domain D on M where φ_0 should be mapped. Hence for each $s > 0$ we can define uniquely $X(t, \theta_s \omega)$ with $X(0, \theta_s \omega) \in D$ lifting the path $\pi X(u, \theta_s \omega)$, $0 \leq u \leq t$ from N to M . Thus if $X(0, \omega) \in D$ is a random point whose distribution is the normalized Riemannian volume on D , then $g(t, \omega) = \text{dist}(x(0, \omega), X(t, \omega))$ is a stationary ergodic process. By the triangle inequality $g(t+s, \omega) \leq g(t, \omega) + g(s, \theta_t \omega)$ since $X(0, \theta_t \omega) = X(t, \omega)$ and $X(s, \theta_t \omega) = X(t+s, \omega)$. By Kingman's subadditive ergodic theorem the limit in (3.4) exists for almost all $x = X(0, \omega)$ in D with probability one. Let x be any point in D and $X(0, \omega) \equiv x$. Then for each $s > 0$,

$$(3.5) \quad \begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \text{dist}(x, X(t, \omega)) &= \lim_{t \rightarrow \infty} \frac{1}{t} \text{dist}(x(s, \omega), X(t, \omega)) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \text{dist}(x(0, \theta_s \omega), X(t-s, \theta_s \omega)) \stackrel{\text{def}}{=} \alpha \end{aligned}$$

with probability one since the distribution of $X(0, \theta_s \omega)$ on D is equivalent to the volume. Thus the limit (3.4) exists, is the same for all $x \in D$, and, by its invariance under isometries, is the same for all $x \in M$.

To show that $\alpha > 0$ it suffices to derive that

$$(3.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} E_x R_x(t) = \alpha > 0.$$

The limit in (3.6) exists by the subadditivity. By Lemma 3.1,

$$(3.7) \quad E_y R_x(1) \geq \text{dist}(x, y) + b_1$$

for some $b_1 > 0$ independent of x and y . Thus by the Markov property of the Brownian motion

$$(3.8) \quad E_x R_x(n) = E_x E_{X(n-1)} R_x(1) \geq E_x R_x(n-1) + b_1.$$

Repeating this argument we obtain $E_x R_x(n) \geq nb_1$ which implies that $\alpha > 0$. \square

3.3. Lemma. *There exists $\kappa \in (0, 1)$ such that for any $x, y \in M$ and $t \geq 1$,*

$$(3.9) \quad P_y\{\kappa s \leq R_x(s) - R_x(0) \leq \kappa^{-1}s \text{ for all } s \geq t\} \geq 1 - \kappa^{-1}e^{-\kappa t}.$$

Proof. Let z_y , r_0 and b_0 be the same as in Lemma 2.1. We may assume that t is big enough. Introduce the following stopping times $\tau_0 = 0$, $\tau_{2j+1} = \inf\{t \geq$

$j: \text{dist}(X(t)), z_{X(j)} \leq r_0/2\}$, $\tau_{2j+2} = \inf\{t \geq \tau_{2j+1}: \text{dist}(X(t), z_{X(j)}) \geq r_0\}$ for $j = 0, 1, \dots$. Denote by A_{j+1} the event $\{j \leq \tau_{2j+1} \leq j + \frac{1}{3} \text{ and } j + \frac{2}{3} \leq \tau_{2j+1} < j + 1\}$ for $j = 0, 1, \dots$. Put

$$\mathcal{A}_l^k = \bigcup_{(l-1)k < j \leq lk} A_j.$$

Then by (3.2),

$$\begin{aligned} R_x(t) &\geq R_x(0) + \sqrt{2}w(t) + \sum_{1 \leq j \leq t} \int_{j-1}^j Q_x(X(s)) ds \\ (3.10) \quad &\geq R_x(0) + \sqrt{2}w(t) + \frac{1}{3}b_0 \sum_{1 \leq j \leq t} \chi_{A_j} \\ &\geq R_x(0) + \sqrt{2}w(t) + \frac{1}{3}b_0 \sum_{1 \leq l \leq [t/k]} \chi_{\mathcal{A}_l^k}, \end{aligned}$$

where $[a]$ denotes the integral part of a . For any $0 < l_1 < l_2 < \dots < l_n \leq [t/k]$, let $\mathcal{E}_{l_1, \dots, l_n}^k$ be the event that all $\mathcal{A}_{l_1}^k, \mathcal{A}_{l_2}^k, \dots, \mathcal{A}_{l_n}^k$ occur and if $l \neq l_i$, $0 < l \leq [t/k]$ for each $i = 1, \dots, n$, then the negation of \mathcal{A}_l^k occurs, i.e.,

$$\mathcal{E}_{l_1, \dots, l_n}^k = \left(\bigcap_{1 \leq i \leq n} \mathcal{A}_{l_i}^k \right) \cap \left(\bigcap_{l \neq l_i} (\Omega \setminus \mathcal{A}_l^k) \right).$$

By the Markov property of the Brownian motion $X(t)$,

$$\begin{aligned} P_y(\mathcal{E}_{l_1, \dots, l_n}^k) &= E_y \chi_{\mathcal{A}_{l_1}^k} E_{X(k)} \chi_{\mathcal{A}_{l_2}^k} \cdots E_{X((L_t-1)k)} \chi_{\mathcal{A}_{l_n}^k} \\ &\leq \left(1 - \inf_{z \in M} P_z(\mathcal{A}_1^k) \right)^{L_t - n}, \end{aligned}$$

where $L_t = [t/k]$ and $\tilde{\mathcal{A}}_l^k = \mathcal{A}_l^k$ if $l = l_i$ for some $i = 1, \dots, n$ and $\tilde{\mathcal{A}}_l^k = \Omega \setminus \mathcal{A}_l^k$ if $l \neq l_i$ for all $i = 1, \dots, n$. Again by the Markov property

$$\begin{aligned} P_z(\mathcal{A}_1^k) &= 1 - P_z \left(\bigcap_{0 < j \leq k} (\Omega \setminus A_j) \right) \\ (3.12) \quad &= 1 - E_z \chi_{\Omega \setminus A_1} E_{X(1)} \chi_{\Omega \setminus A_1} \cdots E_{X(k-1)} \chi_{\Omega \setminus A_1} \\ &\geq 1 - \left(1 - \inf_{v \in M} P_v(A_1) \right)^k. \end{aligned}$$

Since $\text{dist}(z_a, a) \leq R_0$ for any $a \in M$, it follows from the strong Markov property that

$$\begin{aligned} P_v(A_1) &\geq \left(\inf_{w \in M} \inf_{u \in B_w(R_0)} P_w \left\{ \tau_u \left(\frac{r_0}{2} \right) < \frac{1}{3} \right\} \right) \\ (3.13) \quad &\times \left(\inf_{w \in M} \inf_{u \in B_w(r_0/2)} \left(P_u \{ \tau_w(r_0) < 1 \} - P_u \left\{ \tau_w(r_0) < \frac{2}{3} \right\} \right) \right). \end{aligned}$$

Since $g(s, w) = P_w\{\tau_u(r) < s\}$ solves the parabolic equation $\partial g / \partial s = \Delta g$ in the geodesic ball $B_u(r)$ of radius r centered at u with initial condition

$g|_{s=0} = 0$ and boundary condition $g|_{S_u(r)} = 1$, it follows that the right-hand side in (3.13) is positive. Hence, if $k \geq k_0$ is big enough, then the right-hand side of (3.12) is bigger than $\frac{7}{8}$, and so by (3.11),

$$(3.14) \quad P_y(\mathcal{E}_{l_1, \dots, l_n}^{k_0}) \leq \left(\frac{1}{8}\right)^{L_t - n}.$$

Let $\mathcal{J}_t = \sum_{1 \leq l \leq L_t} \chi_{\mathcal{A}_l^{k_0}}$. Then by (3.14),

$$\begin{aligned} P_y\{\mathcal{J}_t \leq T\} &\leq \sum_{0 \leq n \leq T} \sum_{l_1, \dots, l_n} P_y(\mathcal{E}_{l_1, \dots, l_n}^{k_0}) \\ &\leq \sum_{0 \leq n \leq T} \binom{L_t}{n} \left(\frac{1}{8}\right)^{L_t - n} \leq 2^{3T - 2L_t}. \end{aligned}$$

Thus by (3.10) and the reflection principle (see [KS, p. 80]) it follows that

$$\begin{aligned} P_y \left\{ R_x(s) - R_x(0) \leq \frac{b_0 s}{36k_0} \text{ for some } s \geq t \right\} \\ &\leq P_y \left\{ \inf_{m < s \leq m+1} (R_x(s) - R_x(0)) \right. \\ &\quad \left. \leq \frac{b_0(m+1)}{36k_0} \text{ for some } m > t-1 \right\} \\ &\leq \sum_{m > t-1} P_y \left\{ \frac{1}{3} b_0 \mathcal{J}_m + \inf_{m < s \leq m+1} (\sqrt{2} w(s)) \leq \frac{b_0(m+1)}{36k_0} \right\} \\ &\leq \sum_{m > t-1} \left(P_y \left\{ \frac{1}{3} b_0 \mathcal{J}_m \leq 2 \frac{b_0(m+1)}{36k_0} \right\} \right. \\ (3.15) \quad &\quad \left. + P \left\{ \inf_{m < s \leq m+1} \sqrt{2} w(s) \leq -\frac{b_0(m+1)}{36k_0} \right\} \right) \\ &\leq \sum_{m > t-1} \left(P_y \left\{ \mathcal{J}_m + \frac{(m+1)}{6k_0} \right\} \right. \\ &\quad \left. + P \left\{ \sup_{m < s \leq m+1} w(s) \geq \frac{b_0(m+1)}{54k_0} \right\} \right) \\ &\leq \sum_{m > t-1} \left(2^{2-k_0^{-1}(m-1)} + 2P \left\{ w(m+1) \geq \frac{b_0(m+1)}{54k_0} \right\} \right) \\ &\leq (2^{k_0^{-1}} - 1)^{-1} 2^{2-k_0^{-1}(t-1)} + \sum_{m > t-1} \frac{108k_0}{b_0} \exp \left(-\frac{b_0^2(m+1)}{2(54k_0)^2} \right) \\ &\leq \frac{1}{2} \kappa^{-1} e^{-\kappa t} \end{aligned}$$

for some $\kappa > 0$, where we use that

$$(3.16) \quad \int_a^\infty e^{-x^2/2s} dx \leq \int_a^\infty e^{-ax/2s} dx \leq \frac{2s}{a} e^{-a^2/2s}.$$

By Bishops's comparison theorem (see [Ch, pp. 71–73]),

$$(3.17) \quad \sup\{Q_x(y) : x, y \in M, \text{dist}(x, y) \geq 1\} = b_2 < \infty,$$

where b_2 depends only on the lower bound of the Ricci curvature. Thus by (3.2)

$$(3.18) \quad R_x(t) - R_x(0) \leq \sqrt{2}w(t) + b_2t + 1, \quad t \leq \tau_x(1).$$

Again by the reflection principle (see [KS, p. 80]) and (3.16),

$$\begin{aligned} P_y\{R_x(s) - R_x(0) \geq 2b_2s \text{ for some } s \geq t\} \\ &\leq P_y\left\{\sup_{m < s \leq m+1} (R_x(s) - R_x(0)) \geq 2b_2m \text{ for some } m > t-1\right\} \\ &\leq \sum_{m > t-1} P\left\{\sqrt{2} \sup_{m < s \leq m+1} w(s) \geq b_2(m-1) - 1\right\} \\ &\leq \sum_{m > t-1} 2P\{\sqrt{2}w(m+1) \geq (b_2(m-1) - 1)\} \\ &\leq 4 \sum_{m > t-1} (m+1)^{1/2} (b_2(m-1) - 1)^{-1} \exp\left(-\frac{(b_2(m-1) - 1)^2}{4(m+1)}\right) \\ &\leq \frac{1}{2} \kappa^{-1} e^{-\kappa t} \end{aligned}$$

for an appropriately chosen $\kappa > 0$ and $t > 2b_2^{-1} + 2$. This, together with (3.16) yields (3.9) provided κ is small enough. \square

3.4. Lemma. *Let $\infty > r_3 > r_2 > r_1 > 0$, $x \in M$, $\xi \in S_x M$, and $y = \text{Exp}_x(r_2 \xi)$. Then*

$$(3.19) \quad P_y\{\tau_x(r_3) > \tau_x(r_1)\} \leq P_y\{\infty > \tau_x(r_1)\} \leq 2\kappa^{-1} \exp\left(-\frac{\kappa}{2}(r_2 - r_1)\right)$$

and for $t \geq \kappa^{-1}(r_3 - r_2)$,

$$(3.20) \quad P_y\{\tau_x(r_3) \geq t\} \leq \kappa^{-1} e^{-\kappa t},$$

where $\kappa \leq \frac{1}{2}$ satisfies (3.9).

Proof. Note firstly that our choice of $\kappa \leq \frac{1}{2}$ means that $\kappa^{-1}e^{-\kappa} > 1$ and hence that (3.9) holds for all t . By (3.2),

$$R_x(t) - R_x(0) \geq \sqrt{2}w(t),$$

and since $R_x(0) = r_2$ we derive by the submartingale inequality (see [Ks, p. 13]),

$$\begin{aligned} (3.21) \quad &P_y\{\tau_x(r_1) \leq \tfrac{1}{2}(r_2 - r_1)\} \\ &= P_y\left\{\inf_{0 \leq s \leq (r_2 - r_1)/2} (R_x(s) - r_2) \leq r_1 - r_2\right\} \\ &\leq P\left\{\inf_{0 \leq s \leq (r_2 - r_1)/2} w(s) \leq \frac{1}{\sqrt{2}}(r_1 - r_2)\right\} \\ &= P\left\{\sup_{0 \leq s \leq (r_2 - r_1)/2} e^{w(s)} \geq \exp\left\{\frac{1}{\sqrt{2}}(r_2 - r_1)\right\}\right\} \\ &\leq e^{-(r_2 - r_1)/4}. \end{aligned}$$

On the other hand, by (3.9), $R_x(t) - R_x(0) \geq \frac{\kappa}{2}(r_2 - r_1)$ for all $t \geq \frac{1}{2}(r_2 - r_1)$, with probability at least $1 - \kappa^{-1} \exp(-\frac{\kappa}{2}(r_2 - r_1))$, and so

$$(3.22) \quad P_y \left\{ \infty > \tau_x(r_1) \geq \frac{1}{2}(r_2 - r_1) \right\} \leq \kappa^{-1} \exp \left(-\frac{\kappa}{2}(r_2 - r_1) \right),$$

which, together with (3.21), yields (3.19) provided $\kappa \leq \frac{1}{2}$. Next, by (3.9) for $\kappa t \geq r_3 - r_2$,

$$(3.23) \quad \begin{aligned} P_y \{ \tau_x(r_3) \geq t \} &\leq P_y \{ R_x(t) - R_x(0) < r_3 - r_2 \} \\ &\leq P_y \{ R_x(t) - R_x(0) < \kappa t \} \leq \kappa^{-1} e^{-\kappa t}, \end{aligned}$$

proving (3.20). \square

The following lemma is similar to Lemma 2.3 of [KL]; it estimates the probability of the “angular” displacement of the Brownian motion.

3.5. Lemma. *There exists $c > 0$ such that if x, ξ, y, r_1, r_2, r_3 are as in Lemma 3.4, $r_1 \geq r_0 \geq 0$, and*

$$\begin{aligned} t_x(l; r_0, r_1) &= \inf \{ \text{dist}_x^{(r_1)}(\text{Exp}_x(r_1\eta), \text{Exp}_x(r_1\xi)) : \\ &\quad \eta, \xi \in S_x M \text{ and } \text{dist}_x^{(r_0)}(\text{Exp}_x(r_0\eta), \text{Exp}_x(r_0\xi)) \geq l \}^{1/2}, \end{aligned}$$

then

$$(3.24) \quad \begin{aligned} Q_y(x; r_0, r_1, r_3; l) &\stackrel{\text{def}}{=} P_y \left\{ \sup_{t: 0 \leq t \leq \tau_x(r_1) \wedge \tau_x(r_3)} \text{dist}_x^{(r_0)}(\text{Exp}_x(r_0\Xi_x(t)), \text{Exp}_x(r_0\xi)) \geq l \right\} \\ &\leq P_y \{ \tau_x(r_1) \wedge \tau_x(r_3) > t_x(l; r_0, r_1) \} \\ &\quad + c^{-1} (t_x(l; r_0, r_1))^2 \exp(-ct_x(l; r_0, r_1)) \end{aligned}$$

for any $l > 0$ such that $t_x(l; r_0, r_1) \geq 1$.

Proof. Put $t_0 = t_x(l; r_0, r_1)$. Then

$$(3.25) \quad Q_y(x; r_0, r_1, r_3, l) \leq P_y \{ \tau_x(r_1) \wedge \tau_x(r_3) > t_0 \} + \tilde{Q}_y,$$

where

$$\begin{aligned} \tilde{Q}_y &= P_y \left\{ \sup_{0 \leq t \leq \tau_x(r_1) \wedge \tau_x(r_3)} \text{dist}_x^{(r_0)}(\text{Exp}_x(r_0\Xi_x(t)), \text{Exp}_x(r_0\xi)) \geq l \right. \\ &\quad \left. \text{and } \tau_x(r_1) \wedge \tau_x(r_3) \leq t_0 \right\}. \end{aligned}$$

The event in the last probability can occur only if the Brownian motion $X(t)$ will cover distance at least t_0^2 in time not exceeding t_0 , since by the Rauch comparison theorem any smooth curve lying entirely outside the ball $B_x(r_1)$ is at least as long as its projection to $S_x(r_1)$ along the geodesic rays through x . This means that $\tau_y(k_0) \leq t_0$ where $k_0 = [t_0^2]$, and so

$$(3.26) \quad \tilde{Q}_y \leq P_y \{ \tau_y(k_0) \leq t_0 \}.$$

Define the random points y_i , $i = 0, \dots, k_0 - 1$, inductively by $y_0 = y$, $y_{i+1} = X(\tau_{y_i}(1))$. Then

$$(3.27) \quad \tau_y(k_0) \geq \sum_{k_0-1 \geq i \geq 0} \tau_{y_i}(1).$$

By Lemma 2.2 from [Ki] there exists a number $c_1 > 0$ such that for any $z \in M$ and $s > 0$,

$$(3.28) \quad P_z\{\tau_z(1) \leq s\} \leq c_1^{-1} \exp(-c_1 s^{-1}).$$

Now by (3.26)–(3.28) and the strong Markov property,

$$\begin{aligned} \tilde{Q}_y &\leq P_y \left\{ \sum_{l_0-1 \geq i \geq 0} \tau_{y_i}(1) \leq t_0 \right\} \\ (3.29) \quad &\leq P_y\{\tau_{y_i}(1) \leq t_0 k_0^{-1} \text{ for some } i = 0, \dots, k_0 - 1\} \\ &\leq \sum_{l_0-1 \geq i \geq 0} E_y P_{y_i}\{\tau_{y_i}(1) \leq t_0 k_0^{-1}\} \leq k_0 \sup_{z \in M} P_z\{\tau_z(1) \leq t_0 k_0^{-1}\} \\ &\leq k_0 c_1^{-1} \exp(-c_1 k_0 t_0^{-1}) \leq c_1^{-1} t_0^2 \exp(-c_1(t_0 - 1)) \end{aligned}$$

implying (3.24). \square

3.6. Lemma. *Let $p(t, x, y)$ denote the heat kernel on M , i.e., the minimal positive fundamental solution of the parabolic equation $\partial u / \partial t = \Delta u$. There exists a number $a > 0$ such that if $\text{dist}(x, y) > 1$ then*

$$(3.30) \quad p(t, x, y) \leq a^{-1} e^{-a \text{dist}(x, y)}.$$

Proof. The argument is the same as in [Ki, p. 204]. Recall that $p(t, x, y)$ is the transition density of the diffusion $X(t)$, and so by the strong Markov property

$$\begin{aligned} (3.31) \quad p(t, x, y) &= E_x \chi_{\tau_y(1) < t} p(t - \tau_y(1), X(\tau_y(1)), y) \\ &\leq C P_x\{\tau_y(1) < t\} \leq C P_x\{\tau_y(1) < \infty\} \end{aligned}$$

for some $C > 0$ which by (3.19) gives (3.30). \square

4. MAIN RESULTS

For $x \in M$ and $\xi \in S_x M$ let $\mathcal{E}_x(\xi, \varphi)$ denote the set of endpoints at $+\infty$ of the geodesics starting at x whose initial velocity makes the angle less than φ with ξ .

The Hausdorff dimension of a set $U \subset S(\infty)$ corresponding to a pole x is given by

$$\begin{aligned} (4.1) \quad Hd_x(U) &= \inf \left\{ \delta \geq 0: \liminf_{\rho \rightarrow 0} \sum_i \varphi_i^\delta = 0 \right\} \\ &= \sup \left\{ \delta \geq 0: \liminf_{\rho \rightarrow 0} \sum_i \varphi_i^\delta = \infty \right\}, \end{aligned}$$

where the infimum inside the brackets is taken over all countable covers of U by the sets

$$V_x(\xi_i, \varphi_i) = \overline{\mathcal{E}_x(\xi_i, \varphi_i)} \cap S(\infty), \quad \xi_i \in S_x M, \quad 0 \leq \varphi_i \leq \rho.$$

The Hausdorff dimension of a probability measure μ on $S(\infty)$ is defined by

$$(4.2) \quad HD_x(\mu) = \inf\{Hd_x(U): U \subset S(\infty) \text{ and } \mu(U) = 1\}.$$

Our main results correspond to the case where M is the universal cover of a compact nonflat surface N of nonpositive curvature which we will assume throughout this section. On the other hand what we really need for our method is that Lemmas 2.1 and 2.2 hold true and if one takes them as assumptions then the proof of Theorem 4.1 goes through without any further preconditions.

By [B], with probability one the limit

$$(4.3) \quad \lim_{t \rightarrow \infty} \Xi_x(t) = \Xi_x(\infty)$$

of the angular part of $X(t)$ exists, which in the two-dimensional case will follow also from the arguments of this section. The harmonic measures $P(x, \cdot)$ on $S(\infty)$ satisfy

$$(4.4) \quad P(x, \Gamma) = P_x\{\Phi_x \Xi_x(\infty) \in \Gamma\}$$

for any Borel $\Gamma \subset S(\infty)$ with the map $\Phi_x: S_x M \rightarrow S(\infty)$ defined in the introduction. In the same way as in [KL] the positivity of Hausdorff dimensions of $P(x, \cdot)$ will follow from the following result.

4.1. Theorem. *There exists a number $\lambda > 0$ and a sequence of numbers $C_k > 0$ such that*

$$(4.5) \quad P_x\{\|\Xi_x(\infty, \omega) - \xi\| \leq \varphi \text{ and } \omega \in \Omega_x^{(k)}\} \leq C_k \varphi^\lambda$$

for any $x \in M$, $\xi \in S_x M$, and $\varphi \geq 0$ where $\Omega_x^{(k)}$, $k = 1, 2, \dots$, is an increasing sequence of events such that $P_x(\bigcup_k \Omega_x^{(k)}) = 1$.

4.2. Corollary. *For any $x \in M$,*

$$(4.6) \quad HD_x(P(x, \cdot)) \geq \lambda.$$

This corollary follows from Theorem 4.1 by a simple argument explained in §3 of [KL], and so we will need to prove only Theorem 4.1 itself. For any $x \in M$, $\zeta \in S_x M$, $t > 0$, and $\rho > 0$, define

$$U_t(x, \xi, \rho) = \bigcup \{B_{z_k}(2\rho) : 0 \leq k \leq [t(2\kappa^{-1} - \frac{1}{2}\kappa)\rho^{-1}] + 1\}$$

where κ is the same as in (3.9), the balls $B_{z_k}(2\rho)$ of radius 2ρ are centered at the points $z_k = \text{Exp}_x((\frac{1}{2}\kappa t + k\rho)\xi)$, and $[\cdot]$ denotes the integral part. The main step in the proof of Theorem 4.1 is the following result (cf. Lemma 3.1 in [KL]).

4.3. Proposition. *For any $\delta > 0$ there exists $q = q(\delta) > 0$ such that*

$$(4.7) \quad P_z \left\{ \sup_{t \geq 0} \text{dist}_x^{(\rho)}(\text{Exp}_x(\rho \Xi_x(t)), \text{Exp}_x(\rho \xi)) \leq (\kappa \rho)^\delta \right\} \geq 1 - q^{-1} \rho^{-q} e^{-qr}$$

for all $x \in M$, $z = (\rho + r, \xi) \in S_x(\rho + r)$, $\xi \in S_x M$, and $\rho \geq 1$, $r \geq 0$, where κ is the same as in (3.9). Furthermore, for all $t > 0$,

$$(4.8) \quad \begin{aligned} & P_x\{X(t) \in U_t(x, \Xi_x(\infty), t^\delta)\} \\ & \geq P_x\{\kappa t \leq R_x(t) \leq \kappa^{-1} t \text{ and} \\ & \quad \text{dist}_x^{(R_x(t))}(\text{Exp}_x(R_x(t)\Xi_x(\infty)), X(t)) \leq t^\delta\} \geq 1 - q^{-1} t^{-q}. \end{aligned}$$

Proof. We first derive (4.7) for $r = 0$. It suffices to do this only for ρ large enough and for $\delta \leq 1$ which we assume in what follows. For $n = 1, 2, \dots$, put $l_k = \frac{1}{2}(\kappa\rho)^{(3/2)\delta}((\kappa\rho)^\delta + k^2)^{-1}$,

$$(4.9) \quad W_n(z) = \left\{ \eta \in S_x M : \text{dist}_x^{(\rho)}(\text{Exp}_x(\rho\xi), \text{Exp}_x(\rho\eta)) \leq \sum_{k=1}^n l_k \right\}$$

and

$$D_n^\varepsilon(z) = \{\text{Exp}_x(u\eta) : \eta \in W_n(z), r_\varepsilon(n-1) \leq u \leq r_\varepsilon(n+1)\}$$

where $r_\varepsilon(0) = \rho - \varepsilon \log \rho$, $r_\varepsilon(1) = \rho$, and $r_\varepsilon(n) = \rho + \varepsilon(n-1) \log \rho + \sum_{k=1}^{n-1} k^{2/\delta}$ for $n > 1$. We choose ε so that $0 < \varepsilon < 1$ and for any $\eta_1, \eta_2 \in S_x M$ and $u > 0$,

$$(4.10) \quad \begin{aligned} & \text{dist}_x^{(u+\varepsilon \log \rho)}(\text{Exp}_x((u+\varepsilon \log \rho)\eta_1), \text{Exp}_x((u+\varepsilon \log \rho)\eta_2)) \\ & \leq (\kappa\rho)^{\delta/4} \text{dist}_x^{(u)}(\text{Exp}_x(u\eta_1), \text{Exp}_x(u\eta_2)). \end{aligned}$$

For $n = 1, 2, \dots$, set $\partial_n^+(z) = D_n^\varepsilon(z) \cap S_x(r_\varepsilon(n+1))$, $\partial_n^-(z) = D_n^\varepsilon(z) \cap S_x(r_\varepsilon(n-1))$, and $\partial_n^\mathcal{E}(z) = \partial D_n^\varepsilon(z) \setminus (\partial_n^+(z) \cup \partial_n^-(z))$, where ∂D denotes the boundary of D . The picture in §3 of [KL] makes the above notation more transparent.

Let $\tau_n = \inf\{t > 0 : X(t) \notin D_n^\varepsilon(z)\}$ be the exit time of the Brownian motion $X(t)$ from $D_n^\varepsilon(z)$. Then for any $y \in \partial_{n-1}^+$ if $n > 1$ and for $y = z$ if $n = 1$ one has

$$(4.11) \quad \begin{aligned} P_y\{X(\tau_n) \notin \partial_n^+\} & \leq P_y\{\tau_x(r_\varepsilon(n+1)) > \tau_x(r_\varepsilon(n-1))\} \\ & + P_y\{X(\tau_n) \in \partial_n^\mathcal{E}(x)\}. \end{aligned}$$

By (3.19),

$$(4.12) \quad \begin{aligned} & \sup_{y \in S_x(r_\varepsilon(n))} P_y\{\tau_x(r_\varepsilon(n+1)) > \tau_x(r_\varepsilon(n-1))\} \\ & \leq \kappa^{-1} \rho^{-\varepsilon\kappa/2} \exp\left(-\frac{\kappa}{2}(n-1)^{2/\delta}\right). \end{aligned}$$

By (4.10) and the definition of $D_1^\varepsilon(z)$,

$$(4.13) \quad P_z\{X(\tau_1) \in \partial_1^\mathcal{E}(z)\} \leq Q_z(x; r_\varepsilon(0), r_\varepsilon(0), r_\varepsilon(2), \frac{1}{4}(\kappa\rho)^{\delta/4})$$

with Q_y defined by (3.24). Furthermore, by the definition of $D_n^\varepsilon(z)$, $n = 2, 3, \dots$, for any $y \in \partial_{n-1}^+$,

$$(4.14) \quad \begin{aligned} & P_y\{X(\tau_n) \in \partial_n^\mathcal{E}(z)\} \\ & \leq Q_y(x; \rho, r_\varepsilon(n-1), r_\varepsilon(n+1), \frac{1}{2}(\kappa\rho)^{(3/2)\delta}((\kappa\rho)^\delta + n^2)^{-1}). \end{aligned}$$

To employ (3.24) we have to estimate $t_x(l; \rho, \rho+u)$ for certain $l > 0$ and $u \geq 0$. Since the curvature is nonpositive then by the Rauch comparison theorem (see [Ch]), geodesics emanating from x diverge at least as fast as in the zero curvature case, i.e., in the Euclidean plane. Thus for all $u > 0$,

$$(4.15) \quad (t_x(l; \rho, \rho+u))^2 \geq (\rho+u)\rho^{-1}l.$$

Put $u(l) = \max(0, 2R_0\rho l^{-1} - \rho)$. For $u = u(l)$, the right-hand side of (4.15) is not less than $2R_0$ and we can use Lemma 2.2 which yields that for any $u > u(l)$,

$$(4.16) \quad (t_x(l; \rho, \rho+u))^2 \geq 2R_0\gamma_0 \exp(\gamma_0(u - u(l))).$$

For l_n defined before in (4.9) we have

$$u(l_n) = \max(0, 4R_0\rho(\kappa\rho)^{-(3/2)\delta}((\kappa\rho)^\delta + n^2) - \rho)$$

and so by (4.16) for $n \geq 3$ and all ρ large enough

$$(4.17) \quad (t_x(l_n; \rho, r_\varepsilon(n-1)))^2 \geq 2R_0\gamma_0\rho^{\varepsilon\gamma_0} \exp\left(\frac{\gamma_0\delta}{2(\delta+2)}(n-2)^{1+(2/\delta)}\right).$$

Clearly, $t_x(l_2; \rho, r_\varepsilon(1)) = l_2^{1/2}$ and $t_x(l_1; r_\varepsilon(0), r_\varepsilon(0)) = l_1^{1/2}$.

Now (3.20), (3.24), (4.9)–(4.14), and (4.17) imply that (cf. (3.18) in [KL])

$$(4.18) \quad P_y\{X(\tau_n) \notin \partial_n^+\} \leq c_2^{-1}\rho^{-c_2}e^{-c_2n}$$

for some constant $c_2 > 0$ independent of $z, y \in M$ and $n \geq 1$ provided $y = z$ if $n = 1$ and $y \in \partial_{n-1}^+$ if $n > 1$. Since $\sum_{n=1}^\infty l_n \leq (\kappa\rho)^\delta$, (4.18) together with the strong Markov property of $X(t)$ yields

$$\begin{aligned} & P_z\{\text{dist}_x^{(\rho)}(\text{Exp}_x(\rho\Xi_x(\infty)), z) \leq (\kappa\rho)^\delta\} \\ & \geq P_z\left\{\sup_{t \geq 0} \text{dist}_x^{(\rho)}(\text{Exp}_x(\rho\Xi_x(t)), z) \leq (\kappa\rho)^\delta\right\} \\ (4.19) \quad & \geq E_z \chi_{X(\tau_1) \in \partial_1^+} E_{X(\tau_1)} \chi_{X(\tau_2) \in \partial_2^+} \cdots \\ & \geq 1 - P_z\{X(\tau_1) \notin \partial_1^+\} - \sum_{n=2}^\infty \sup_{y \in \partial_{n-1}^+} P_y\{X(\tau_n) \notin \partial^+\} \\ & \leq 1 - c_2^{-2}\rho^{-c_2} \end{aligned}$$

implying (4.7) for $r = 0$.

Now let $r > 0$. Then using (4.19) with $\rho + r$ instead of ρ we obtain by Lemma 2.2 that

$$\begin{aligned} & P_z\left\{\sup_{t \geq 0} \text{dist}_x^{(\rho)}(\text{Exp}_x(\rho\Xi_x(t)), \text{Exp}_x(\rho\xi)) \leq (\kappa\rho)^\delta\right\} \\ (4.20) \quad & \geq P_z\left\{\sup_{t \geq 0} \text{dist}_x^{(\rho+r)}(\text{Exp}_x((\rho+r)\Xi_x(t)), z) \leq (\kappa\rho)^\delta \gamma e^{\gamma r}\right\} \\ & \geq 1 - c_2^{-2}\gamma^{-c_2}(\kappa\rho)^{-c_2\delta}e^{-c_2\gamma r}, \end{aligned}$$

provided $(\kappa\rho)^\delta \geq 2R_0$ which we may assume to be true. Thus (4.7) follows for any $r \geq 0$. Finally, (3.9) and (4.19) together with the Markov property of $X(t)$ yield

$$\begin{aligned} & P_x\{X(t) \in U_t(x, \Xi_x(\infty), t^\delta)\} \\ & \geq P_x\{\kappa t \leq R_x(t) \leq \kappa^{-1}t \text{ and } \text{dist}_x^{(R_x(t))}(\text{Exp}_x(R_x(t)\Xi_x(\infty)), X(t)) \leq t^\delta\} \\ & \geq \int_{B_x(\kappa^{-1}t) \setminus B_x(\kappa t)} P(t, x, dz) \\ & \quad \times P_z\{\text{dist}_x^{(\rho_x(z))}(\text{Exp}_x(\rho_x(z)\Xi_x(\infty)), z) \leq (\kappa\rho_x(z))^\delta\} \\ (4.21) \quad & \geq (1 - c_2^{-2}(\kappa t)^{-c_2})(1 - \kappa^{-1}e^{-\kappa t}) \end{aligned}$$

implying (4.8) where $\rho_x(z) = \text{dist}(x, z)$ and $P(t, x, \Gamma) = P_x\{X(t) \in \Gamma\}$. \square

Using (3.30) Theorem 4.1 follows from Proposition 4.3 proceeding verbatim as at the end of §3 in [KL]. \square

Consider the measures $\mu_x = \Phi_x m_x$ introduced in §1. If M is the universal cover of a compact surface N of nonpositive curvature such that the geodesic flow on N is ergodic then the stable and unstable foliations are absolutely continuous and we conclude that all the measures μ_x are equivalent. All the measures $P(x, \cdot)$ are equivalent by the Harnack inequality.

4.4. Theorem. *If the geodesic flow on N is ergodic then the measures μ_x and $P(x, \cdot)$ on $S(\infty)$ are singular unless the curvature is constant.*

Proof. Let

$$(4.22) \quad \beta = -\lim_{t \rightarrow \infty} \frac{1}{t} \log p(t, x, X(t)) = \lim_{\delta \rightarrow 0} \liminf_{t \rightarrow \infty} \frac{1}{t} N(x, \delta, t),$$

where $N(x, \delta, t)$ is the minimal cardinality of a finite set E satisfying $P_x\{\text{dist}(X(t), E) \leq 1\} \geq 1 - \delta$. The main step in the proof of Theorem 4.4 which is similar to [L] is to show that if the measures μ_x and $P(x, \cdot)$ are equivalent, then $\beta = \alpha\gamma_1$, where α is from (3.4) and γ_1 is from Lemma 2.4. We will need the following proposition.

4.5. Proposition. *With probability 1*

$$(4.23) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \text{dist}(X(t), \text{Exp}_x(\alpha t \Xi_x(\infty))) = 0.$$

Proof. By (3.9), (4.7) and the Markov property of $X(t)$ we have

$$(4.24) \quad \begin{aligned} & P_x\{\text{dist}^{(R_x(t))}(\text{Exp}_x(R_x(t)\Xi_x(t+\varepsilon t)), \text{Exp}_x(R_x(t)\Xi_x(\infty))) > t^\delta\} \\ & \leq P_x\{\text{dist}^{(R_x(t))}(\text{Exp}_x(R_x(t)\Xi_x(t+\varepsilon t)), \text{Exp}_x(R_x(t)\Xi_x(\infty))) \\ & \quad > (\kappa R_x(t))^\delta \text{ and } \kappa t \leq R_x(t) \leq \kappa^{-1}t\} + \kappa^{-1}e^{-\kappa t} \\ & = \int_{B_x(\kappa^{-1}t) \setminus B_x(\kappa t)} P(t, x, dy) \int_M P(\varepsilon t, y, dz) \\ & \quad \times P_z\{\text{dist}^{(\rho_x(y))}(\text{Exp}_x(\rho_x(y)\xi_x(z)), \text{Exp}_x(\rho_x(y)\Xi_x(\infty))) \\ & \quad > (\kappa \rho_x(y))^\delta\} + \kappa^{-1}e^{-\kappa t} \\ & \leq \int_{B_x(\kappa^{-1}t) \setminus B_x(\kappa t)} P(t, x, dy) \int_{\{z : \kappa \varepsilon t \leq \rho_x(z) - \rho_x(y) \leq \kappa^{-1}\varepsilon t\}} P(\varepsilon t, y, dz) \\ & \quad \times P_z\left\{\sup_{s \geq 0} \text{dist}^{(\rho_x(y))}(\text{Exp}_x(\rho_x(y)\xi_x(z)), \right. \\ & \quad \left. \text{Exp}_x(\rho_x(y)\Xi_x(s))) > (\kappa \rho_x(y))^\delta\right\} \\ & \quad + \kappa^{-1}e^{-\kappa t} + \kappa^{-1}e^{-\kappa \varepsilon t} \leq q^{-1}(\kappa t)^{-q}e^{-q\kappa \varepsilon t} + \kappa^{-1}e^{-\kappa t} + \kappa^{-1}e^{-\kappa \varepsilon t}, \end{aligned}$$

where $\xi_x(z) \in S_x M$ and $z = \text{Exp}_x(\rho_x(z)\xi_x(z))$. Thus by the Borel-Cantelli lemma for almost all ω there exists $N_1(\omega)$ such that for all $n \geq N_1(\omega)$,

$$(4.25) \quad \text{dist}^{(R_x(n, \omega))}(\text{Exp}_x(R_x(n, \omega)\Xi_x(n+\varepsilon n, \omega), \text{Exp}_x(R_x(n, \omega)\Xi_x(\infty, \omega))) \leq n^\delta.$$

It follows from (3.4) that for almost all ω there exists $N_2(\omega)$ such that for all $n \geq N_2(\omega)$,

$$(4.26) \quad \begin{aligned} (1-\varepsilon)\alpha & \leq n^{-1}R_x(n, \omega) \leq (1+\varepsilon)\alpha \quad \text{and} \\ \alpha & \leq n^{-1}R_x(n+\varepsilon n, \omega) \leq (1+2\varepsilon)\alpha. \end{aligned}$$

Thus for all $n \geq N_3(\omega) = \max(N_1(\omega), N_2(\omega))$,

$$(4.27) \quad \text{dist}(X(n + \varepsilon n, \omega), \text{Exp}_x(\alpha n \Xi_x(\infty, \omega))) \leq n^\delta + 4\varepsilon \alpha n.$$

Furthermore, by (3.16), (3.18), the reflection principle, and the Markov property of $X(t)$,

$$\begin{aligned} & P_x \left\{ \sup_{0 \leq u \leq s} \text{dist}(X(n), X(n+u)) > l \right\} \\ &= \int_M P(n, x, dy) P_y \left\{ \sup_{0 \leq u \leq s} \text{dist}(y, X(u)) > l \right\} \\ (4.28) \quad & \leq \sup_{y \in M} P_y \left\{ \sup_{0 \leq u \leq s} R_y(u) > l \right\} \\ & \leq P \left\{ \sup_{0 \leq u \leq s} w(u) > l - b_2 s - 1 \right\} = 2P\{w(s) > l - b_2 s - 1\} \\ & \leq \frac{2\sqrt{s}}{(l - b_2 s - 1)} \exp \left(-\frac{1}{2s}(l - b_2 s - 1) \right). \end{aligned}$$

We use (4.28) first with $s = \varepsilon n$ and $l = 2b_2\varepsilon n + 1$ and then with $s = 1$ and $l = \varepsilon n$. For those values inequality (4.28) together with the Borel-Cantelli lemma yield that for almost all ω there exists $N_4(\omega)$ such that for all $n \geq N_4(\omega)$,

$$(4.29) \quad \text{dist}(X(n, \omega), X(n + \varepsilon n, \omega)) \leq 2b_2\varepsilon n + 1$$

and

$$(4.30) \quad \sup_{0 \leq u \leq 1} \text{dist}(X(n), X(n+u)) \leq \varepsilon n.$$

Since $\varepsilon > 0$ arbitrary, by taking in (4.27) any positive $\delta < 1$ we obtain (4.23) from (4.27), (4.29) and (4.30). Note that our proof, unlike [L], does not use directly the fact that M is the universal cover of a compact manifold and it requires only (3.4) which may hold true in other circumstances, as well (see, for instance [Pi, §V]). \square

The remaining part of the proof of Theorem 4.4 proceeds in the same way as in [L] employing the conformal equivalence of M and the unit hyperbolic disk, the fact that conformal maps transform the Brownian motion into the time changed Brownian motion and certain inequalities connecting the numbers α , β and γ_1 before and after a conformal transformation. The proof of inequalities (a) and (b) on p. 192 in [L] remains the same in our case and inequality (c) based on Corollary 2.16 from [Ka1] also holds true due to the ergodicity of the geodesic flow on N . \square

4.5. Remark. Note that for the universal cover of a compact negatively curved surface S , [KL] asserts that if the Hausdorff dimension of the harmonic measures is equal to one then the curvature is constant. This was proved by showing that the harmonic measures give rise to a certain measure invariant with respect to the geodesic flow on S which is a Gibbs state corresponding to some Hölder continuous function and if the above Hausdorff dimension is one, then Pesin's formula holds true for this measure. By uniqueness, this measure must coincide with the Liouville measure. Therefore, the harmonic measure class is the

same as geodesic measure class and by [L] the curvature is constant. In the nonpositive curvature case the uniqueness of an invariant measure satisfying Pesin's formula is not known, and we are not able to conclude that if the Hausdorff dimension is one for harmonic measures then the surface has constant curvature.

4.6. *Remark.* The method of the present paper works as well if one wants to describe the Martin boundary of M with $\dim M = 2$ in the same way as in [Ki]. It is necessary to adjust Lemmas 3.1, 4.1, and 4.2 from [Ki] in the spirit of the modifications we needed in §§3 and 4 here as compared to the negative curvature case in [Ki and KL]. In the two-dimensional case M is conformally equivalent to the hyperbolic disk D and hence the Laplace-Beltrami operator Δ_M on M equals $\sigma\Delta_D$, where σ is a positive function and Δ_D is the Laplace-Beltrami operator on D . Thus the harmonic functions, in particular the minimal positive harmonic functions, on M and D are essentially the same. However, the notion of the Martin boundary ∂M involves not only minimal positive harmonic functions, but also the topology induced by Martin's functions $k_y(x) = g(x, y)/g(x_0, y)$, where $g(x, y) = \int_0^\infty p(t, x, y) dt$ is the Green's function. This topology is defined by saying that a sequence of points $y_i \in M$ is fundamental if the sequence k_{y_i} converges to a harmonic function. Fundamental sequences are equivalent if their limiting functions coincide, and ∂M is the set of equivalence classes. The topology on $M \cup \partial M$ is defined by the metric based on functions k_y (see (4.7) in [Ki]). Since the Green's function g on M has no explicit connections with the Green's function on D , then, in general, the Martin topology on M cannot be obtained via the conformal equivalence of M and D . Thus it still makes sense to go all the way of [Ki] with the modifications as in the present paper even when $\dim M = 2$ to derive that the Martin boundary ∂M with the Martin topology is homeomorphic to a circle. The identification of the space of minimal positive harmonic functions on M with a circle via the conformal equivalence of M and D corresponds, in general, to a different topology. Of course, the identification of the Martin boundary is more interesting in the full multidimensional generality of [B] where one cannot use the conformal equivalence, and so no description of the space of minimal positive harmonic functions is available at all.

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